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NORMAL CRITERIA FOR FAMILIES OF MEROMORPHIC FUNCTIONS

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Abstract

By using Nevanlinna theory, we prove some normality criteria for a family of meromorphic functions under a condition on differential polynomials generated by the members of the family.

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1 Introduction

Let D be a domain in the complex plane $\mathbb{C}$ and $\mathcal{F}$ be a family of meromorphic functions in D. The family $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $\{f_v\} \subset \mathcal{F}$, there exists a subsequence $\{f_{v_i}\}$ such that $\{f_{v_i}\}$ converges spherically locally uniformly in D, to a meromorphic function or ∞.

In 1989, Schwick proved:

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Theorem A ([6], Theorem 3.1)**.** *Let k, n be positive integers such that $n \geq k+3$. Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D such that for every $f \in \mathcal{F}$, $(f^n)^{(k)}(z) \neq 1$ for all $z \in D$. Then $\mathcal{F}$ is normal on D.*

Theorem B ([6], Theorem 3.2)**.** *Let k, n be positive integers such that $n \geq k+1$. Let $\mathcal{F}$ be a family of entire functions in a complex domain D such that for every $f \in \mathcal{F}$, $(f^n)^{(k)}(z) \neq 1$ for all $z \in D$. Then $\mathcal{F}$ is normal on D.*

The following normality criterion was established by Pang and Zalcman [7] in 1999:

Theorem C ([7])**.** *Let n and k be natural numbers and $\mathcal{F}$ be a family of holomorphic functions in a domain D all of whose zeros have multiplicity at least k. Assume that $f^n f^{(k)} - 1$ is non-vanishing for each $f \in \mathcal{F}$. Then $\mathcal{F}$ is normal in D.*

The main purpose of this paper is to establish some normality criteria for the case of more general differential polynomials. Our main results are as follows:

Theorem 1. *Take q $(q \geq 1)$ distinct nonzero complex values $a_1, \ldots, a_q$, and q positive integers (or $+\infty$) $\ell_1, \ldots \ell_q$. Let n be a nonnegative integer, and let $n_1, \ldots, n_k, t_1, \ldots, t_k$ be positive integers ($k \geq 1$). Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D such that for every $f \in \mathcal{F}$ and for every $m \in \{1, \ldots, q\}$, all zeros of $f^n(f^{n_1})^{(t_1)} \cdots (f^{n_k})^{(t_k)} - a_m$ have multiplicity at least ℓ_m. Assume that*

a) $n_j \geq t_j$ *for all* $1 \leqslant j \leqslant k$, *and* $\ell_i \geq 2$ *for all* $1 \leqslant i \leqslant q$,

b) $\sum_{i=1}^{q} \frac{1}{\ell_i} < \frac{qn-2+\sum_{j=1}^{k} q(n_j-t_j)}{n+\sum_{j=1}^{k}(n_j+t_j)}$.

Then $\mathcal{F}$ is a normal family.

Take $q = 1$ and $\ell_1 = +\infty$, we get the following corollary of Theorem 1:

Corollary 2. *Let a be a nonzero complex value, let n be a nonnegative integer, and $n_1, \ldots, n_k, t_1, \ldots, t_k$ be positive integers. Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D such that for every $f \in \mathcal{F}$, $f^n(f^{n_1})^{(t_1)} \cdots (f^{n_k})^{(t_k)} - a$ is nowhere vanishing on D. Assume that*

a) $n_j \geq t_j$ *for all* $1 \leqslant j \leqslant k$,

b) $n + \sum_{j=1}^{k} n_j \geq 3 + \sum_{j=1}^{k} t_j$.

Then $\mathcal{F}$ is normal on D.

We remark that in the case where $n \geq 3$, condition $a)$ in the above corollary implies condition $b)$; and in the case where $n = 0$ and $k = 1$, Corollary 2 gives Theorem A.

For the case of entire functions, we shall prove the following result:

Theorem 3. *Take q ($q \geq 1$) distinct nonzero complex values $a_1, \ldots, a_q$, and q positive integers (or $+\infty$) $\ell_1, \ldots \ell_q$. Let n be a nonnegative integer, and let $n_1, \ldots, n_k, t_1, \ldots, t_k$ be positive integers ($k \geq 1$). Let $\mathcal{F}$ be a family of holomorphic functions in a complex domain D such that for every $f \in \mathcal{F}$ and for every $m \in \{1, \ldots, q\}$, all zeros of $f^n (f^{n_1})^{(t_1)} \cdots (f^{n_k})^{(t_k)} - a_m$ have multiplicity at least ℓ_m. Assume that*
a) $n_j \geq t_j$ for all $1 \leqslant j \leqslant k$, and $\ell_i \geq 2$ for all $1 \leqslant i \leqslant q$,
b) $\sum_{i=1}^{q} \frac{1}{\ell_i} < \frac{qn - 1 + \sum_{j=1}^{k} q(n_j - t_j)}{n + \sum_{j=1}^{k} n_j}$.
Then $\mathcal{F}$ is a normal family.

Take $q = 1$ and $\ell_1 = +\infty$, Theorem 3 gives the following generalization of Theorem B, except for the case $n = k + 1$. So for the latter case, we add a new proof of Theorem B in the Appendix which is slightly simpler than the original one.

Corollary 4. *Let a be a nonzero complex value, let n be a nonnegative integer, and $n_1, \ldots, n_k, t_1, \ldots, t_k$ be positive integers. Let $\mathcal{F}$ be a family of holomorphic functions in a complex domain D such that for every $f \in \mathcal{F}$, $f^n (f^{n_1})^{(t_1)} \cdots (f^{n_k})^{(t_k)} - a$ is nowhere vanishing on D. Assume that*
a) $n_j \geq t_j$ for all $1 \leqslant j \leqslant k$,
b) $n + \sum_{j=1}^{k} n_j \geq 2 + \sum_{j=1}^{k} t_j$.
Then $\mathcal{F}$ is normal on D.

In the case where $n \geq 2$, condition $a)$ in the above corollary implies condition $b)$.

Remark 5. *Our above results remain valid if the monomial $f^n (f^{n_1})^{(t_1)} \cdots (f^{n_k})^{(t_k)}$ is replaced by the following polynomial*

$$f^n (f^{n_1})^{(t_1)} \cdots (f^{n_k})^{(t_k)} + \sum_I c_I f^{n_I} (f^{n_{1I}})^{(t_{1I})} \cdots (f^{n_{kI}})^{(t_{kI})},$$

where c_I is a holomorphic function on D, and n_I, n_{jI}, t_{jI} are nonnegative integers satisfying

$$\alpha_I := \frac{\sum_{j=1} t_{jI}}{n_I + \sum_{j=1}^{k} n_{jI}} < \alpha := \frac{\sum_{j=1} t_j}{n + \sum_{j=1}^{k} n_j}.$$

2 Some notations and results of Nevanlinna theory

Let ν be a divisor on $\mathbb{C}$. The counting function of ν is defined by

$$N(r,\nu) = \int_1^r \frac{n(t)}{t}dt \quad (r > 1), \text{ where } n(t) = \sum_{|z|\le t} \nu(z).$$

For a meromorphic function f on $\mathbb{C}$ with $f \not\equiv \infty$, denote by ν_f the pole divisor of f, and the divisor $\overline{\nu}_f$ is defined by $\overline{\nu}_f(z) := \min\{\nu_f(z), 1\}$. Set $N(r,f) := N(r,\nu_f)$ and $\overline{N}(r,f) := N(r,\overline{\nu}_f)$.
The proximity function of f is defined by

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta,$$

where $\log^+ x = \max\{\log x, 0\} \quad$ for $x \geq 0$.
The characteristic function of f is defined by

$$T(r,f) := m(r,f) + N(r,f).$$

We state the Lemma on Logarithmic Derivative, the First and Second Main Theorems of Nevanlinna theory.

Lemma on Logarithmic Derivative. *Let f be a nonconstant meromorphic function on $\mathbb{C}$, and let k be a positive integer. Then the equality*

$$m(r, \frac{f^{(k)}}{f}) = o(T(r,f))$$

holds for all $r \in [1,\infty)$ excluding a set of finite Lebesgue measure.

First Main Theorem. *Let f be a meromorphic functions on $\mathbb{C}$ and a be a complex number. Then*

$$T(r, \frac{1}{f-a}) = T(r,f) + O(1).$$

Second Main Theorem. *Let f be a nonconstant meromorphic function on $\mathbb{C}$. Let $a_1, \ldots, a_q$ be q distinct values in $\mathbb{C}$. Then*

$$(q-1)T(r,f) \leqslant \overline{N}(r,f) + \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f-a_i}) + o(T(r,f)),$$

for all $r \in [1, \infty)$ excluding a set of finite Lebesgue measure.

3 Proof of our results

To prove our results, we need the following lemmas:

Lemma 6 (Zalcman's Lemma, see [8]). *Let $\mathcal{F}$ be a family of meromorphic functions defined in the unit disc $\triangle$. Then if $\mathcal{F}$ is not normal at a point $z_0 \in \triangle$, there exist, for each real number α satisfying $-1 < \alpha < 1$,*

1) *a real number r, $0 < r < 1$,*
2) *points z_n, $|z_n| < r$, $z_n \to z_0$,*
3) *positive numbers $\rho_n, \rho_n \to 0^+$,*
4) *functions f_n, $f_n \in \mathcal{F}$*

such that

$$g_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n^{\alpha}} \to g(\xi)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function and $g^{\#}(\xi) \leqslant g^{\#}(0) = 1$. Moreover, the order of g is not greater than 2. Here, as usual, $g^{\#}(z) = \frac{|g'(z)|}{1+|g(z)|^2}$ is the spherical derivative.

Lemma 7 (see [2]). *Let g be a entire function and M is a positive constant. If $g^{\#}(\xi) \leqslant M$ for all $\xi \in \mathbb{C}$, then g has order at most one.*

Remark 8. *In Lemma 6, if $\mathcal{F}$ is a family of holomorphic functions, then by Hurwitz theorem, g is a holomorphic function. Therefore, by Lemma 7, the order of g is not greater than 1.*

We consider a nonconstant meromorphic function g in the complex plane $\mathbb{C}$, and its first p derivatives. A differential polynomial P of g is defined by

$$P(z) := \sum_{i=1}^{n} \alpha_i(z) \prod_{j=0}^{p} (g^{(j)}(z))^{S_{ij}},$$

where S_{ij} $(1 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant p)$ are nonnegative integers, and $\alpha_i \not\equiv 0$ $(1 \leqslant i \leqslant n)$ are small (with respect to g) meromorphic functions. Set

$$d(P) := \min_{1 \leqslant i \leqslant n} \sum_{j=0}^{p} S_{ij} \text{ and } \theta(P) := \max_{1 \leqslant i \leqslant n} \sum_{j=0}^{p} jS_{ij}.$$

In 2002, J. Hinchliffe [5] generalized theorems of Hayman [3] and Chuang [1] and obtained the following result:

Proposition 9. *Let g be a transcendental meromorphic function, let P(z) be a non-constant differential polynomial in g with $d(P) \geq 2$. Then*

$$T(r,g) \leqslant \frac{\theta(P)+1}{d(P)-1}\overline{N}(r,\frac{1}{g}) + \frac{1}{d(P)-1}\overline{N}(r,\frac{1}{P-1}) + o(T(r,g)),$$

for all $r \in [1,+\infty)$ excluding a set of finite Lebesgues measure.

In order to prove our results, we now give the following generalization of the above result:

Lemma 10. *Let $a_1, \dots, a_q$ be distinct nonzero complex numbers. Let g be a nonconstant meromorphic function, let P(z) be a nonconstant differential polynomial in g with $d(P) \geq 2$. Then*

$$T(r,g) \leqslant \frac{q\theta(P)+1}{qd(P)-1}\overline{N}(r,\frac{1}{g}) + \frac{1}{qd(P)-1}\sum_{j=1}^{q}\overline{N}(r,\frac{1}{P-a_j}) + o(T(r,g)),$$

for all $r \in [1,+\infty)$ excluding a set of finite Lebesgues measure.
Moreover, in the case where g is a entire function, we have

$$T(r,g) \leqslant \frac{q\theta(P)+1}{qd(P)}\overline{N}(r,\frac{1}{g}) + \frac{1}{qd(P)}\sum_{j=1}^{q}\overline{N}(r,\frac{1}{P-a_j}) + o(T(r,g)),$$

for all $r \in [1,+\infty)$ excluding a set of finite Lebesgue measure.

Proof. For any z such that $|g(z)| \leqslant 1$, since $\sum_{j=0}^{p} S_{ij} \geq d(P)$ $(1 \leqslant i \leqslant n)$, we have

$$\begin{aligned}\frac{1}{|g(z)|^{d(P)}} &= \frac{1}{|P(z)|}\cdot\frac{|P(z)|}{|g(z)|^{d(P)}}\\ &\leqslant \frac{1}{|P(z)|}\cdot\sum_{i=1}^{n}\big(|\alpha_i(z)|\prod_{j=0}^{p}|\frac{g^{(j)}(z)}{g(z)}|^{S_{ij}}\big).\end{aligned}$$

This implies that for all $z \in \mathbb{C}$,

$$\log^+ \frac{1}{|g(z)|^{d(P)}} \leqslant \log^+ \big(\frac{1}{|P(z)|} \cdot \sum_{i=1}^{n} \big(|\alpha_i(z)| \prod_{j=0}^{p} |\frac{g^{(j)}(z)}{g(z)}|^{S_{ij}}\big)\big).$$

Therefore, by the Lemma on Logarithmic Derivative and by the First Main Theorem, we have

$$\begin{aligned} d(P)m(r, \frac{1}{g}) \leqslant m(r, \frac{1}{P}) + o(T(r,g)) &= T(r, \frac{1}{P}) - N(r, \frac{1}{P}) + o(T(r,g)) \\ &= T(r, P) - N(r, \frac{1}{P}) + o(T(r,g)). \end{aligned}$$

On the other hand, by the Second Main Theorem (used with the $q+1$ different values $0, a_1, ..., a_q$) we have

$$qT(r,P) \leqslant \overline{N}(r,P) + \overline{N}(r, \frac{1}{P}) + \sum_{j=1}^{q} \overline{N}(r, \frac{1}{P - a_j}) + o(T(r,g)),$$

Hence,

$$\begin{aligned} d(P)m(r, \frac{1}{g}) \leqslant \frac{1}{q}\big(\overline{N}(r,P) + \overline{N}(r, \frac{1}{P}) + \sum_{j=1}^{q} \overline{N}(r, \frac{1}{P - a_j})\big) \\ - N(r, \frac{1}{P}) + o(T(r,g)). \end{aligned}$$

Therefore, by the First Main Theorem, we have

$$\begin{aligned} d(P)T(r,g) &= d(P)T(r, \frac{1}{g}) + O(1) \\ &= d(P)m(r, \frac{1}{g}) + d(P)N(r, \frac{1}{g}) + O(1) \\ &\leqslant \frac{1}{q}\big(\overline{N}(r,P) + \overline{N}(r, \frac{1}{P}) + \sum_{j=1}^{q} \overline{N}(r, \frac{1}{P - a_j})\big) \\ &\quad + d(P)N(r, \frac{1}{g}) - N(r, \frac{1}{P}) + o(T(r,g)). \end{aligned} \tag{3.1}$$

We have

$$\frac{1}{g^{d(P)}} = \frac{1}{P(z)} \sum_{i=1}^{n} \big(\alpha_i g^{(\sum_{j=0}^{p} S_{ij}) - d(P)} \prod_{j=0}^{p} (\frac{g^{(j)}}{g})^{S_{ij}}\big).$$

(note that $(\sum_{j=0}^{p} S_{ij}) - d(P) \geq 0$). Therefore,

$$
\begin{aligned}
d(P)\nu_{\frac{1}{g}} &\leqslant \nu_{\frac{1}{P}} + \max_{1\leqslant i\leqslant n}\{\nu_{\alpha_i} + \sum_{j=0}^{p} jS_{ij}\overline{\nu}_{\frac{1}{g}}\} \\
&\leqslant \nu_{\frac{1}{P}} + \sum_{i=1}^{n} \nu_{\alpha_i} + \theta(P)\overline{\nu}_{\frac{1}{g}},
\end{aligned}
$$

where ν_ϕ is the pole divisor of the meromorphic ϕ and $\overline{\nu}_\phi := \min\{\nu_\phi, 1\}$. This implies,

$$
d(P)\nu_{\frac{1}{g}} - \nu_{\frac{1}{P}} + \frac{1}{q}\overline{\nu}_{\frac{1}{P}} \leqslant (\theta(P) + \frac{1}{q})\overline{\nu}_{\frac{1}{g}} + \sum_{i=1}^{n} \nu_{\alpha_i},
$$

(note that for any z_0, if $\nu_{\frac{1}{g}}(z_0) = 0$ then $d(P)\nu_{\frac{1}{g}}(z_0) - \nu_{\frac{1}{P}}(z_0) + \frac{1}{q}\overline{\nu}_{\frac{1}{P}}(z_0) \leqslant 0$). Then,

$$
\begin{aligned}
d(P)N(r, \frac{1}{g}) - N(r, \frac{1}{P}) + \frac{1}{q}\overline{N}(r, \frac{1}{P}) &\leqslant (\theta(P) + \frac{1}{q})\overline{N}(r, \frac{1}{g}) + \sum_{i=1}^{n} N(r, \alpha_i) \\
&= (\theta(P) + \frac{1}{q})\overline{N}(r, \frac{1}{g}) + o(T(r, g)).
\end{aligned}
$$

Combining with (3.1), we have

$$
d(P)T(r, g) \leqslant \frac{1}{q}\big(\overline{N}(r, P) + \sum_{j=1}^{q} \overline{N}(r, \frac{1}{P - a_j})\big) + (\theta(P) + \frac{1}{q})\overline{N}(r, \frac{1}{g}) + o(T(r, g)).
$$

On the other hand, by the definition of the differential polynomial P, $\mathrm{Pole}(P) \subset \cup_{i=1}^{n} \mathrm{Pole}(\alpha_i) \cup \mathrm{Pole}(g)$. Hence (since $\overline{N}(r, \alpha_i) \leq T(r, \alpha_i) = o(T(r, g)$ for $i = 1, ..., n$), we get

$$
\begin{aligned}
d(P)T(r, g) &\leqslant \frac{1}{q}\big(\overline{N}(r, g) + \sum_{j=1}^{q} \overline{N}(r, \frac{1}{P - a_j})\big) + (\theta(P) + \frac{1}{q})\overline{N}(r, \frac{1}{g}) + o(T(r, g)) \\
&\leqslant \frac{1}{q}\big(T(r, g) + \sum_{j=1}^{q} \overline{N}(r, \frac{1}{P - a_j})\big) + (\theta(P) + \frac{1}{q})\overline{N}(r, \frac{1}{g}) + o(T(r, g)).
\end{aligned}
\tag{3.2}
$$

Therefore,

$$T(r,g) \leqslant \frac{q\theta(P)+1}{qd(P)-1}\overline{N}(r,\frac{1}{g}) + \frac{1}{qd(P)-1}\sum_{j=1}^{q}\overline{N}(r,\frac{1}{P-a_j}) + o(T(r,g)).$$

In the case where g is an entire function, the first inequality in (3.2) becomes

$$d(P)T(r,g) \leqslant \frac{1}{q}\sum_{j=1}^{q}\overline{N}(r,\frac{1}{P-a_j}) + (\theta(P)+\frac{1}{q})\overline{N}(r,\frac{1}{g}) + o(T(r,g)).$$

This implies that

$$T(r,g) \leqslant \frac{\theta(P)q+1}{qd(P)})\overline{N}(r,\frac{1}{g}) + \frac{1}{qd(P)}\sum_{j=1}^{q}\overline{N}(r,\frac{1}{P-a_j}) + o(T(r,g)).$$

We have completed the proof of Lemma 10. □

Proof of Theorem 1. Without loss the generality, we may asssume that D is the unit disc. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. By Lemma 6, for $\alpha = \frac{\sum_{j=1}^{k} t_j}{n+\sum_{j=1}^{k} n_j}$ there exist

1) a real number $r,\ 0 < r < 1$,
2) points $z_v,\ |z_v| < r,\ z_v \to z_0$,
3) positive numbers $\rho_v, \rho_v \to 0^+$,
4) functions $f_v,\ f_v \in \mathcal{F}$

such that

$$g_v(\xi) = \frac{f_v(z_v+\rho_v\xi)}{\rho_v^{\alpha}} \to g(\xi) \tag{3.3}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function and $g^{\#}(\xi) \leqslant g^{\#}(0) = 1$.
On the other hand,

$$\begin{aligned}\big(g_v^{n_j}(\xi)\big)^{(t_j)} &= \big((\frac{f_v(z_v+\rho_v\xi)}{\rho_v^{\alpha}})^{n_j}\big)^{(t_j)}\\ &= \frac{1}{\rho_v^{n_j\alpha-t_j}}(f_v^{n_j})^{(t_j)}(z_v+\rho_v\xi).\end{aligned}$$

Therefore, by the definition of α and by (4.1), we have

$$f_v^n(z_v+\rho_v\xi)(f_v^{n_1})^{(t_1)}(z_v+\rho_v\xi)\cdots(f_v^{n_k})^{(t_k)}(z_v+\rho_v\xi)$$
$$= g_v^n(\xi)(g_v^{n_1}(\xi))^{(t_1)}\dots(g_v^{n_k}(\xi))^{(t_k)} \to g^n(\xi)(g^{n_1}(\xi))^{(t_1)}\dots(g^{n_k}(\xi))^{(t_k)} \tag{3.4}$$

spherically uniformly on compact subsets of $\mathbb{C}$.

Now, we prove the following claim:

Claim: $g^n(\xi)(g^{n_1}(\xi))^{(t_1)}\dots(g^{n_k}(\xi))^{(t_k)}$ *is non-contstant.*

Since g is non-constant and $n_j \geq t_j$ $(j=1,\dots,k)$, it easy to see that $(g^{n_j}(\xi))^{(t_j)} \not\equiv 0$, for all $j \in \{1,\dots,k\}$. Hence, $g^n(\xi)(g^{n_1}(\xi))^{(t_1)}\dots(g^{n_k}(\xi))^{(t_k)} \not\equiv 0$.

Suppose that $g^n(\xi)(g^{n_1}(\xi))^{(t_1)}\dots(g^{n_k}(\xi))^{(t_k)} \equiv a$, $a \in \mathbb{C}\setminus\{0\}$. We first remark that, from conditions $a), b)$, we have that in the case $n=0$, there exists $i \in \{1,\dots,k\}$ such that $n_i > t_i$. Therefore, in both cases ($n=0$ and $n \neq 0$), since $a \neq 0$, it is easy to see that g is entire having no zero. So, by Lemma 7, $g(\xi)=e^{c\xi+d}$, $c \neq 0$. Then

$$\begin{aligned} g^n(\xi)(g^{n_1}(\xi))^{(t_1)}\cdots(g^{n_k}(\xi))^{(t_k)} &= e^{nc\xi+nd}(e^{n_1c\xi+n_1d})^{(t_1)}\cdots(e^{n_kc\xi+n_kd})^{(t_k)} \\ &= (n_1c)^{t_1}\cdots(n_kc)^{t_k}e^{(n+\sum_{j=1}^k n_j)c\xi+(n+\sum_{j=1}^k n_j)d}. \end{aligned}$$

Then $(n_1c)^{t_1}\cdots(n_kc)^{t_k}e^{(n+\sum_{j=1}^k n_j)c\xi+(n+\sum_{j=1}^k n_j)d} \equiv a$, which is impossible. So, $g^n(\xi)(g^{n_1}(\xi))^{(t_1)}\dots(g^{n_k}(\xi))^{(t_k)}$ is nonconstant, which proves the claim.

By the assumption of Theorem 1 and by Hurwitz's theorem, for every $m \in \{1,\dots,q\}$, all zeros of $g(\xi)^n(g^{n_1}(\xi))^{(t_1)}\cdots(g^{n_k}(\xi))^{(t_k)}-a_m$ have multiplicity at least ℓ_m.

For any $j \in \{1,\cdots,k\}$, we have that $(g^{n_j}(\xi))^{(t_j)}$ is nonconstant. Indeed, if $(g^{n_j}(\xi))^{(t_j)}$ is constant for some $j \in \{1,\dots,k\}$, then since $n_j \geq t_j$, and since g is nonconstant, we get that $n_j=t_j$ and $g(\xi)=a\xi+b$, where a,b are constants, $a \neq 0$. Thus, we can write

$$g(\xi)^n(g^{n_1}(\xi))^{(t_1)}\cdots(g^{n_k}(\xi))^{(t_k)} = c(a\xi+b)^{n+\sum_{j=1}^k(n_j-t_j)},$$

where c is a nonzero constant. This contradicts to the fact that all zeros of $g(\xi)^n(g^{n_1}(\xi))^{(t_1)}\cdots(g^{n_k}(\xi))^{(t_k)}-a_m$ have multiplicity at least $\ell_m \geq 2$ (note that $a_m \neq 0$, and that, by condition b) of Theorem 1, $n+\sum_{j=1}^k(n_j-t_j)>0$). Thus, $(g^{n_j}(\xi))^{(t_j)}$ is nonconstant, for all $j \in \{1,\cdots,k\}$.

On the other hand, we can write

$$(g^{n_j})^{(t_j)} = \sum c_{m_0,m_1,\ldots,m_{t_j}} g^{m_0}(g')^{m_1}\ldots(g^{(t_j)})^{m_{t_j}},$$

$c_{m_0,m_1,\ldots,m_{t_j}}$ are constants, and $m_0, m_1, \ldots, m_{t_j}$ are nonnegative integers such that $m_0+\cdots+m_{t_j}=n_j, \sum_{j=1}^{t_j} jm_j = t_j$. Thus, by an easy computation, we get that $d(P)=n+\sum_{j=1}^k n_j, \theta(P)=\sum_{j=1}^k t_j$.

Now, we apply Lemma 10 for the differential polynomial

$$P = g(\xi)^n(g^{n_1}(\xi))^{(t_1)}\cdots(g^{n_k}(\xi))^{(t_k)}.$$

By Lemma 10, we have (note that, by condition b) of Theorem 1, $n+\sum_{j=1}^k n_j \geq 2$)

$$\begin{aligned} T(r,g) \leqslant & \frac{q\sum_{j=1}^k t_j + 1}{qn+q\sum_{j=1}^k n_j - 1}\overline{N}(r,\frac{1}{g}) \\ & + \frac{1}{qn+q\sum_{j=1}^k n_j - 1}\sum_{m=1}^q \overline{N}(r,\frac{1}{P-a_m}) + o(T(r,g)). \end{aligned} \tag{3.5}$$

For any $m\in\{1,\ldots,q\}$, we have, by the First Main Theorem,

$$\begin{aligned} \overline{N}(r,\frac{1}{P-a_m}) &= \overline{N}(r,\frac{1}{g^n(g^{n_1})^{(t_1)}\cdots(g^{n_k})^{(t_k)}-a_m}) \\ &\leqslant \frac{1}{\ell_m}N(r,\frac{1}{g^n(g^{n_1})^{(t_1)}\cdots(g^{n_k})^{(t_k)}-a_m}) \\ &\leqslant \frac{1}{\ell_m}T(r,g^n(g^{n_1})^{(t_1)}\cdots(g^{n_k})^{(t_k)}) + O(1) \\ &= \frac{1}{\ell_m}m(r,g^n(g^{n_1})^{(t_1)}\cdots(g^{n_k})^{(t_k)}) \\ &\quad + \frac{1}{\ell_m}N(r,g^n(g^{n_1})^{(t_1)}\cdots(g^{n_k})^{(t_k)}) + O(1). \end{aligned} \tag{3.6}$$

By the Lemma on Logarithmic Derivative and by the First Main Theorem,

$$\begin{aligned}
m(r,&g^n(g^{n_1})^{(t_1)}\cdots(g^{n_k})^{(t_k)})+N(r,g^n(g^{n_1})^{(t_1)}\cdots(g^{n_k})^{(t_k)})\\
&\leqslant m(r,\frac{g^n(g^{n_1})^{(t_1)}\cdots(g^{n_k})^{(t_k)}}{g^n g^{n_1}\cdots g^{n_k}})+m(r,g^n g^{n_1}\cdots g^{n_k})\\
&\qquad +N(r,g^n(g^{n_1})^{(t_1)}\cdots(g^{n_k})^{(t_k)})\\
&\leqslant (n+\sum_{j=1}^k n_j)m(r,g)+N(r,g^n(g^{n_1})^{(t_1)}\cdots(g^{n_k})^{(t_k)})+o(T(r,g))\\
&=(n+\sum_{j=1}^k n_j)m(r,g)+(n+\sum_{j=1}^k n_j)N(r,g)+(\sum_{j=1}^k t_j)\overline{N}(r,g)+o(T(r,g))\\
&\leqslant (n+\sum_{j=1}^k n_j)T(r,g)+(\sum_{j=1}^k t_j)\overline{N}(r,g)+o(T(r,g)).
\end{aligned}\tag{3.7}$$

Combining with (3.6), for all $m\in\{1,\dots,q\}$ we have

$$\begin{aligned}
\overline{N}(r,\frac{1}{P-a_m})&\leqslant\frac{1}{\ell_m}(n+\sum_{j=1}^k n_j)T(r,g)+\frac{1}{\ell_m}(\sum_{j=1}^k t_j)\overline{N}(r,g)+o(T(r,g))\\
&\leq\frac{1}{\ell_m}(n+\sum_{j=1}^k n_j+\sum_{j=1}^k t_j)T(r,g)+o(T(r,g)).
\end{aligned}\tag{3.8}$$

Therefore, by (3.5) and by the First Main Theorem, we have

$$\begin{aligned}
(qn+q\sum_{j=1}^k n_j-1)T(r,g)&\leqslant(q\sum_{j=1}^k t_j+1)\overline{N}(r,\frac{1}{g})+\sum_{m=1}^q\overline{N}(r,\frac{1}{P-a_m})+o(T(r,g))\\
&\leqslant(q\sum_{j=1}^k t_j+1)T(r,g)+(n+\sum_{j=1}^k n_j+\sum_{j=1}^k t_j)(\sum_{m=1}^q\frac{1}{\ell_m})T(r,g)+o(T(r,g).
\end{aligned}$$

This implies that

$$\frac{qn+\sum_{j=1}^k q(n_j-t_j)-2}{n+\sum_{j=1}^k(n_j+t_j)}T(r,g)\leqslant\sum_{m=1}^q\frac{1}{\ell_m}T(r,g)+o(T(r,g)).$$

Combining with assumption $b)$ we get that g is constant. This is a contradiction. Hence $\mathcal{F}$ is a normal family. We have completed the proof of Theorem 1. $\square$

We can obtain Theorem 3 by an argument similar to the the proof of Theorem 1: We first remark that although condition b) of Theorem 3 is different from condition b) of Theorem 1, whereever it has been used in the proof of Theorem 1 before equation (3.5), the condition b) of Theorem 3 still allows the same conclusion. And from equation (3.5) on we modify as follows : Since $\mathcal{F}$ is a family of holomorphic functions and by Remark 8, g is an entire functions. So, similarly to (3.5), by Lemma 10, we have

$$\begin{aligned}T(r,g) &\leqslant \frac{q\sum_{j=1}^{k} t_j + 1}{qn + q\sum_{j=1}^{k} n_j}\overline{N}(r,\frac{1}{g}) + \frac{1}{q(n+\sum_{j=1}^{k} n_j)}\sum_{m=1}^{q}\overline{N}(r,\frac{1}{P-a_m}) + o(T(r,g))\\ &\leqslant \frac{q\sum_{j=1}^{k} t_j + 1}{qn + q\sum_{j=1}^{k} n_j}T(r,g) + \frac{1}{q(n+\sum_{j=1}^{k} n_j)}\sum_{m=1}^{q}\overline{N}(r,\frac{1}{P-a_m}) + o(T(r,g)).\end{aligned} \tag{3.9}$$

Since g is a holomorphic function, $\overline{N}(r,g) = 0$. Therefore, by (3.6) and (3.7) (which remain unchanged), we have

$$\overline{N}(r,\frac{1}{P-a_m}) \leqslant \frac{1}{\ell_m}(n+\sum_{j=1}^{k} n_j)T(r,g) + o(T(r,g)). \tag{3.10}$$

By (3.9), (3.10), we have

$$\frac{qn + \sum_{j=1}^{k} q(n_j - t_j) - 1}{n + \sum_{j=1}^{k} n_j}T(r,g) \leqslant \sum_{m=1}^{q}\frac{1}{\ell_m}T(r,g) + o(T(r,g)).$$

Combining with assumption $b)$ of Theorem 3, we get that g is constant. This is a contradiction. We have completed the proof of Theorem 3. □

In connection with Remark 5, we note that the proofs of Theorem 1 and Theorem 3 remain valid for the case where the monomial $f^n(f^{n_1})^{(t_1)}\cdots(f^{n_k})^{(t_k)}$ is replaced by the following polynomial

$$f^n(f^{n_1})^{(t_1)}\cdots(f^{n_k})^{(t_k)} + \sum_I c_I f^{n_I}(f^{n_{1I}})^{(t_{1I})}\cdots(f^{n_{kI}})^{(t_{kI})},$$

where c_I is a holomorphic function on D, and n_I, n_{jI}, t_{jI} are nonnegative integers satisfying

$$\alpha_I := \frac{\sum_{j=1} t_{jI}}{n_I + \sum_{j=1}^{k} n_{jI}} < \alpha := \frac{\sum_{j=1} t_j}{n + \sum_{j=1}^{k} n_j}.$$

In fact, since $\alpha_I < \alpha$ and by (4.1), we get

$$g_{I_v}(\xi) := \frac{f_v(z_v + \rho_v \xi)}{\rho_v^{\alpha_I}} = \rho_v^{\alpha - \alpha_I} g_v(\xi) \to 0,$$

spherically uniformly on compact subsets of $\mathbb{C}$.
Therefore, similarly to (3.4)

$$\begin{aligned} c_I(z_v + \rho_v\xi) f_v^{n_I}(z_v + \rho_v\xi)(f_v^{n_{1I}})^{(t_{1I})}(z_v + \rho_v\xi) \cdots (f_v^{n_{kI}})^{(t_{kI})}(z_v + \rho_v\xi) \\ = c_I(z_v + \rho_v\xi) g_{I_v}^{n_I}(\xi)(g_v^{n_{1I}}(\xi))^{(t_{1I})} \ldots (g_{I_v}^{n_{Ik}}(\xi))^{(t_{kI})} \to 0, \end{aligned}$$

spherically uniformly on compact subsets of $\mathbb{C}$.
This implies that

$$\begin{aligned} & f_v^n(z_v + \rho_v\xi)(f_v^{n_1})^{(t_1)}(z_v + \rho_v\xi) \cdots (f_v^{n_k})^{(t_k)}(z_v + \rho_v\xi) \\ & \quad + \sum_I c_I(z_v + \rho_v\xi) f_v^{n_I}(z_v + \rho_v\xi)(f_v^{n_{1I}})^{(t_{1I})}(z_v + \rho_v\xi) \cdots (f_v^{n_{kI}})^{(t_{kI})}(z_v + \rho_v\xi) \\ & \quad = g_v^n(\xi)(g_v^{n_1}(\xi))^{(t_1)} \ldots (g_v^{n_k}(\xi))^{(t_k)} \\ & \qquad + \sum_I c_I(z_v + \rho_v\xi) g_{I_v}^{n_I}(\xi)(g_{I_v}^{n_{1I}}(\xi))^{(t_{1I})} \ldots (g_{I_v}^{n_{Ik}}(\xi))^{(t_{kI})} \\ & \quad \to g^n(\xi)(g^{n_1}(\xi))^{(t_1)} \ldots (g^{n_k}(\xi))^{(t_k)}. \end{aligned} \tag{3.11}$$

spherically uniformly on compact subsets of $\mathbb{C}$.

We use again the proofs of Theorem 1 and Theorem 3 for the general case above after changing (3.4) by (3.11). □

4 Appendix

Using our methods above, we give a slightly simpler proof of the case of Theorem B above which did not follow from our Corollary 4:

Theorem 11 ([6], Theorem 3.2, case $n = k + 1$)**.** *Let k be a positive integer and a be a nonzero constant. Let $\mathcal{F}$ be a family of entire functions in a complex domain D such that for every $f \in \mathcal{F}$, $(f^{k+1})^{(k)}(z) \neq a$ for all $z \in D$. Then $\mathcal{F}$ is normal on D.*

In order to prove the above theorem we need the following lemma:

Lemma 12 ([4]). *Let g be a transcendental holomorphic function on the complex plane $\mathbb{C}$, and k be a positive integer. Then $(g^{k+1})^{(k)}$ assumes every nonzero value infinitely often.*

Proof of Theorem 11. Without loss the generality, we may assume that D is the unit disc. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. Then, by Lemma 6, for $\alpha = \frac{k}{k+1}$ there exist

1) a real number $r,\ 0 < r < 1$,
2) points $z_v,\ |z_v| < r,\ z_v \to z_0$,
3) positive numbers $\rho_v, \rho_v \to 0^+$,
4) functions $f_v,\ f_v \in \mathcal{F}$

such that

$$g_v(\xi) = \frac{f_v(z_v + \rho_v \xi)}{\rho_v^{\alpha}} \to g(\xi) \tag{4.1}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant holomorphic function and $g^{\#}(\xi) \leqslant g^{\#}(0) = 1$.
Therefore

$$\begin{aligned}(f_v^{k+1})^{(k)}(z_v + \rho_v \xi) &= \big(\big(\frac{f_v(z_v + \rho_v \xi)}{\rho_v^{\alpha}}\big)^{k+1}\big)^{(k)} \\ &= \big(g_v^{k+1}(\xi)\big)^{(k)} \to (g^{k+1}(\xi))^{(k)}\end{aligned}$$

spherically uniformly on compact subsets of $\mathbb{C}$.

By Hurwitz's theorem either $(g^{k+1})^{(k)} \equiv a$, either $(g^{k+1})^{(k)} \neq a$. On the other hand, it is easy to see that there exists z_0 such that $(g^{k+1})^{(k)}(z_0) = a$ (the case where g is a nonconstant polynomial is trivial and the case where g is transcendental follows from Lemma 12). Hence, $(g^{k+1})^{(k)} \equiv a$. Therefore g has no zero point. Hence, by Lemma 7, $g(\xi) = e^{c\xi + d},\ c \neq 0$. Then $a \equiv (g^{k+1})^{(k)}(\xi) \equiv ((k+1)c)^k e^{(k+1)(c\xi+d)}$, which is impossible. □

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